JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 52, 142–150 (1975)

Dynamic Programming and the Optimization of Two-Point Boundary Value Systems

NESTOR DISTEFANO

University of California, Berkeley, California 94720

Submitted by Richard Bellman

1. INTRODUCTION

In this paper we deal with the optimization of systems whose behavior is governed by generally nonlinear difference equations subject to two-point boundary value conditions. Problems of this kind arise naturally in connection with the optimal design of a number of engineering systems. Here we shall present the ideas in the context of problems in structural mechanics although the method may be applied to similar processes in other engineering fields.

It is shown that the optimization of a structural chain formed by a number of blocks, $2N$-dimensional each, is governed by the solution of two coupled N-dimensional functional equations. This formulation is of interest in the applications in view of the fact that a standard dynamic programming treatment of the same problem would demand the solution of a $2N$-dimensional functional equation. An alternative, less demanding algorithm in terms of storage capacity, may be still implemented if the structure is linearly elastic. Finally, an example involving the optimization of a continuous beam is presented to illustrate the application of the method.

2. FORMULATION OF THE PROBLEM

We consider a serial structural system formed by a number of blocks forming a chain as indicated in Fig. 1. The system will be described by a state vector $(u_i v_i)$, where u_i and v_i are N-dimensional vectors denoting

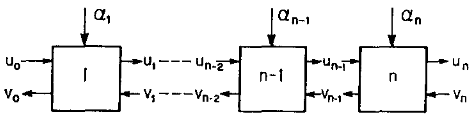

FIGURE 1.

a physical characteristic of the system at the interface i, between blocks i and $i+1$. To fix ideas, and with no loss in generality, we shall assume that v_i and u_i are vectors of generalized forces and displacements, respectively. To simplify the presentation we shall also assume that the dimension of vectors u_i and v_i is N, independent of i. This restriction may be easily removed, if desired.

The behavior of the system is assumed to be governed by a two-point boundary value difference system such as

$$\begin{aligned} u_{i-1} &= H_i(u_i, v_i, \alpha_i), \qquad & u_0 &= a, \\ v_{i-1} &= G_i(u_i, v_i, \alpha_i), \qquad & u_n &= b, \end{aligned} \tag{2.1}$$

where H_i and G_i are generally nonlinear vector functions and α_i is an unknown vector that denotes the design of the block i, to be determined by conditions of optimality. To this end we associate a cost C_i to the ith block which, with no loss in generality, may be considered a function of the type

$$C_i = C_i(u_{i-1}, u_i, \alpha_i). \tag{2.2}$$

Engineering considerations usually dictate one or more constraints of the type

$$g_i(u_{i-1}, u_i, \alpha_i) < 0. \tag{2.3}$$

Combination of (2.2) and (2.3) result in an effective cost function defined: C_i is given by (2.2) if (2.3) is satisfied, and $C_i = \infty$, otherwise.

The problem is now to find a design $\{\alpha_i\}$, $i = 1, 2,..., n$, for the chains of n blocks such that the additive cost $\sum_{i=1}^{n} C_i$ is minimized. Here, in trying a direct dynamic programming solution, we recognize a first difficulty. In fact, the boundary valueness of the problem leads us to the determination of a minimum value function in terms of the data values u_0 and u_n, i.e.,

$$f_n(u_0, u_n) = \min_{\alpha_i} \sum_{i=1}^{n} C_i. \tag{2.4}$$

It is not apparent however, how to construct a functional equation for $f_n(u_0, u_n)$ that results in a useful computational device. This difficulty may be removed at the expense of enlarging the imbedding family such as to include in the minimum value function, the missing components of the state, say v_n, assuming that they are known. We then optimize this larger family of problems using a standard dynamic programming solution and recover finally the original problem by choosing the elements of this larger family that yield the minimum value function. This automatically furnishes the missing boundary conditions of the optimal design. This method has been described in [1]. Let us briefly see now how this idea applies in the present problem. We introduce the minimum value function

$$\bar{f}_n(u_n, v_n) = \min_{\alpha_i} \sum_{i=1}^{n} C_i(u_{i-1}, u_i, \alpha_i), \tag{2.5}$$

where the explicit dependence of $\bar{f}_n$ on u_0 has been omitted. Standard arguments in dynamic programming lead immediately to the functional equation:

$$\bar{f}_n(u_n, v_n) = \min_{\alpha_n}[C_n(u_{n-1}, u_n, \alpha_n) + \bar{f}_{n-1}(u_{n-1}, v_{n-1})], \tag{2.6}$$

subject to

$$\begin{aligned} u_{n-1} &= H_n(u_n, v_n, \alpha_n), \\ v_{n-1} &= G_n(u_n, v_n, \alpha_n), \end{aligned} \tag{2.7}$$

and the initial condition

$$\bar{f}_1(u_1, v_1) = \min_{\alpha_1} C_1[H_1(u_1, v_1, \alpha_1), u_1, \alpha_1]. \tag{2.8}$$

The optimal design of the chain is obtained in the backward direction by using the policy function

$$\alpha_i^{\text{opt}} = \alpha_i(u_i, v_i), \tag{2.9}$$

where u_i and v_i satisfy the difference equations (2.7) subject to the end conditions

$$\begin{aligned} u_n &= b, \\ v_n &= \arg\min_{v} \bar{f}_n(b, v). \end{aligned} \tag{2.10}$$

The difficulty with the solution just outlined lies in the fact that the associated functional equation is $2N$-dimensional. Thus, for very modest values of N we may easily reach the limits of a feasible numerical solution. In the following sections, we discuss some ways to reduce the dimensionality of the numerical algorithm. This will be done by exploiting adroitly some special features of the process being discussed.

3. Optimization of Systems of Variational Origin

In most cases of importance in the applications, the behavior of the system to be optimized is governed by a variational principle. This is particularly true in connection with nonlinear elastic structural systems, our present framework for discussion.

Let $E_i(u_{i-1}, u_i, \alpha_i)$ be the potential energy of the ith block with a given design α_i, in terms of displacements. Then the determination of the equilibrium configuration of a system of n blocks with design $\{\alpha_i\}$, $i = 1, 2,..., n$, reduces to the determination of the displacement field $\{u_i\}$ such as to minimize the potential energy function

$$E = \sum_{i=1}^{n} E_i(u_{i-1}, u_i, \alpha_i). \tag{3.1}$$

Let $g_n(u_n)$ be the minimum potential energy of a system of n blocks, i.e.,

$$g_n(u_n) = \min_{u_i} \sum_{i=1}^{n} E_i(u_{i-1}, u_i, \alpha_i). \tag{3.2}$$

It is clear that $g_n(u_n)$ satisfies the functional equation

$$g_n(u_n) = \min_{u_{n-1}}[E_n(u_{n-1}, u_n, \alpha_n) + g_{n-1}(u_{n-1})], \tag{3.3}$$

subject to the initial condition

$$g_1(u_1) = E_1(a, u_1, \alpha_1). \tag{3.4}$$

The quantity that minimizes in (3.3), a function of u_n and α_n, will be denoted by u_{n-1}^*, i.e.,

$$u_{n-1}^*(u_n, \alpha_n) = \arg\min_{u_{n-1}}[E_n(u_{n-1}, u_n, \alpha_n) + g_{n-1}(u_{n-1})]. \tag{3.5}$$

Introducing now the optimum value function

$$\begin{aligned} h_n(u_n) &= \text{minimum cost of the structural system of } n \text{ blocks,} \\ &\quad \text{with } u_0 = a, \end{aligned} \tag{3.6}$$

we can construct the functional relationship

$$h_n(u_n) = \min_{\alpha_n}[C_n(u_{n-1}^*, u_n, \alpha_n) + h_{n-1}(u_{n-1}^*)], \tag{3.7}$$

subject to

$$h_1(u_1) = \min_{\alpha_1} C_1(a, u_1, \alpha_1). \tag{3.8}$$

The optimal design of the nth block of the system is now given by the quantity $\alpha_n{}^*$ that minimizes in (3.7), i.e.,

$$\alpha_n{}^*(u_n) = \arg\min_{\alpha_n}[C_n(u_{n-1}^*, u_n, \alpha_n) + h_{n-1}(u_{n-1}^*)], \tag{3.9}$$

where u_{n-1}^*, the displacement vector that provides the elastic equilibrium between the nth block and the rest of the structure, is given by (3.5).

The method just outlined provides the solution of our structural optimization problem as the solution of a coupled system of two N-dimensional functional equations, namely Eqs. (3.3)–(3.4) and (3.7)–(3.8). This solution clearly contrasts with that outlined in Section 2 which requires the solution of a $2N$-dimensional functional equation. This is, of course, of interest on computational grounds. In the next section, we explore the consequences of assuming linear structural behavior in the solution of the present system of functional equations.

4. Optimization of Linear Structural Systems I

When the system to be optimized is linear, we can further advance the analytical treatment of Eqs. (3.3)–(3.4), a fact that results in additional computational efficiency. In effect, if the system is linear, the potential energy stored in the nth block is a quadratic function of the form

$$\begin{aligned} E_n(u_{n-1}, u_n, \alpha_n) &= (u_{n-1}, K_{11}(\alpha_n)u_{n-1}) + (u_n, K_{22}(\alpha_n)u_n) \\ &\quad + 2(u_{n-1}, K_{12}(\alpha_n)u_n) + 2(u_{n-1}, p_{1n}) + 2(u_n, p_{2n}), \end{aligned} \tag{4.1}$$

where (K_{ij}), $i, j = 1, 2$, a symmetric positive definite $2N$-matrix, is the *stiffness matrix* of the nth block, and (p_{in}), $i = 1, 2$, is a loading vector [2]. Then, if E_n given by (4.1) is substituted in (3.3), it is well known that the policy function u_{n-1}^* given by (3.5) will be linear in u_n, i.e.,

$$u_{n-1}^* = R_n u_n + r_n, \tag{4.2}$$

where the matrix R_n and vector v_n are functions of α_n that satisfy initial value difference equations of the form

$$\begin{aligned} R_n &= Q_n(R_{n-1}, \alpha_n), & R_1 &= 0, \\ r_n &= q_n(r_{n-1}, R_{n-1}, \alpha_n), & r_1 &= a, \end{aligned} \tag{4.3}$$

where Q_n and q_n are discrete Riccati operators whose precise form is not relevant for our present purposes.

Now, since R_{n-1} and r_{n-1} depend on α_{n-1}^*, which in turn depends on u_{n-1}^*, it is clear that Eqs. (4.2)–(4.3) furnish u_{n-1}^* in an implicit manner. Its numerical determination may be done by a direct search procedure as follows. Using the table of policy functions (3.9) of the dynamic program (3.7)–(3.8), we determine the optimal sequence $\{\alpha_i(u_{n-1}^*)\}$, $i = n-1, n-2,..., 1$, of the structure with $n-1$ blocks, associated to every point in the u_{n-1}^* grid. In this fashion, we may recursively compute the quantities $R_n(u_{n-1}^*)$ and $q_n(u_{n-1}^*)$ by means of Eqs. (4.3). Substitution of these quantities in (4.2) provides a means for the determination of all possible solutions of the Eqs. (4.2)–(4.3). The accuracy of this solution depends on the coarseness of the u_{n-1}^* grid. A refinement of the estimates so obtained may be easily done by iteration.

5. Optimization of Linear Structural Systems II

In general, there is no need to derive Eqs. (4.2)–(4.3) from a variational principle. In fact, when the system is linear, we may always assume a linear transformation like (4.2) and derive recurrence equations for R_n and r_n using ideas of invariant imbedding. This method shares the advantage to apply to all linear systems, regardless whether they derive from a variational principle or not. We illustrate these ideas in the example of the next section.

6. Application to the Optimization of an Elastic Continuous Beam

In order to show how the ideas of the preceding sections may be applied to specific structural problems, we present an example involving the optimization of an elastic continuous beam such as shown in Fig. 2.

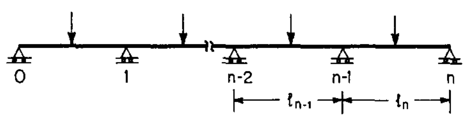

FIGURE 2.

To simplify the presentation we shall assume linear beam behavior, and piecewise constant moment of inertia I_i, $i = 1, 2,..., n$. In addition, we assume first that the span lengths l_i are given in advance.

Now, if ϕ_i denotes the rotation at the ith support, standard structural theory leads to the following difference equation in terms of displacements

$$\alpha_i\phi_{i-1} + 2(\alpha_i + \alpha_{i+1})\phi_i + \alpha_{i+1}\phi_{i+1} = -(m_i^{(2)} + m_{i+1}^{(1)}), \tag{6.1}$$

where $\alpha_i = 2EI_i/l_i$ is the stiffness coefficient of the $(i-1, i)$ span, and $m_i^{(1)}$ and $m_i^{(2)}$ are the fixed end moments at the left and the right of the $(i-1, i)$ span, respectively. As usual, E denotes the modulus of elasticity. We assume the boundary conditions

$$\phi_0 = a, \qquad \phi_n = b, \tag{6.2}$$

and introduce the Riccati transformation

$$\phi_{n-1} = R_n \phi_n + r_n . \tag{6.3}$$

Substitution of (6.3) in (6.1) leads, after simple manipulations, to the following recurrence relations for R_n and r_n

$$\begin{aligned} R_n &= \frac{\alpha_n}{\alpha_{n-1}R_{n-1} + 2(\alpha_{n-1} + \alpha_n)}, \qquad R_1 = 0, \\ r_n &= \frac{\alpha_{n-1}r_{n-1} + m_{n-1}^{(2)} + m_n^{(1)}}{\alpha_{n-1}R_{n-1} + 2(\alpha_{n-1} + \alpha_n)}, \qquad r_1 = a. \end{aligned} \tag{6.4}$$

For illustration purposes each span will be assumed to be constrained both in stress and deflection. The deflection constraint at a point j of the ith beam is given by

$$\frac{2}{\alpha_i l_i} \int_0^{l_i} M_i \bar{M}_i^{(j)} \, dx \leqslant \delta_j , \tag{6.5}$$

where δ_j is a given quantity, $\bar{M}_i^{(j)}$ is the bending moment due to a unit load applied at the point j on the simple beam, and M_i is the bending moment due to external forces and end rotations given by

$$\begin{aligned} M_i(x) = M^0(x) &+ m_i^{(1)} + \alpha_i(2\phi_{i-1} + \phi_i) \\ &+ [(m_i^{(2)} - m_i^{(1)} + \alpha_i(\phi_i - \phi_{i-1}))/l_i]x. \end{aligned} \tag{6.6}$$

In Eq. (6.6), M^0 denotes the bending moment of the external loads acting on the simple beam, and x is the independent variable measured from the support $i - 1$.

The stress constraint at a point z of the ith beam is given by the inequality

$$| M_i(x_z)|/S_i \leqslant \sigma, \tag{6.7}$$

where σ is the allowable stress and S_i is the section modulus of the beam.

In the applications, the design variables $\alpha_i(k)$ and $S_i(k)$ will depend upon an integer variable $k = 1, 2, \dots, K$, denoting all beam sections available. An effective cost C_i may be now defined for the ith beam with section k:

$$C_i = \begin{cases} C_i(\phi_{i-1}, \phi_i, k), & \text{if } \ (6.5)\text{–}(6.7) \text{ are satisfied} \\ \infty, & \text{otherwise,} \end{cases} \tag{6.8}$$

where $C_i(\phi_{i-1}, \phi_i, k)$ is a given function denoting the cost of the ith beam of length l_i with section k, including the cost of the joints at the ith and $(i-1)$th supports. The specific form of the function C_i is not relevant for our present purposes.

Thus, if $f_n(\phi_n)$ denotes the minimum cost of the continuous beam with n spans, subject to a rotation ϕ_n at the nth support,

$$f_n(\phi_n) = \min_k [C_n(\phi_{n-1}, \phi_n, k) + f_{n-1}(\phi_{n-1})], \tag{6.9}$$

subject to the initial condition

$$f_1(\phi_1) = \min_k C_i(a, \phi_1, k), \tag{6.10}$$

where ϕ_{n-1} is given by the Eqs. (6.3)–(6.4) in the manner described in Section 4. Clearly, ϕ_{n-1} depends on k through its dependence with α_n. In this fashion, we are able to handle the optimization of an elastic continuous beam by means of a one-dimensional dynamic programming procedure.

It is worth noting that we may introduce in this formulation other design variables of interest at the expense, of course, of increasing the dimensionality of the functional equation. Thus, for example, if $f_n(\phi_n, L_n)$ denotes the minimum cost of the continuous beam with n spans of lengths l_i, $i = 1$ to n, such that $l_1 + l_2 + \cdots + l_n = L_n$ subject to a rotation ϕ_n at the nth support, we have

$$f_n(\phi_n, L_n) = \min_{k, l_n} [C_n(\phi_{n-1}, \phi_n, k, l_n) + f_{n-1}(\phi_{n-1}, L_n - l_n)], \tag{6.11}$$

subject to the initial condition

$$f_1(\phi_1 l_1) = \min_k C_1(a, \phi_1, k, l_1), \tag{6.12}$$

where, for clarity, we have explicitly indicated the dependence of the cost C_n on the length l_n of the beam.

7. Discussion

It has been shown that the optimization of $2N$-dimensional nonlinear structural systems may be cast as the solution of two coupled N-dimensional functional equations. The presentation was made assuming that the structure is formed by a chain of blocks having the same dimensionality N. This is not an essential restriction. Extension to variable dimensionality, together with studies on numerical feasibility, will be the subject matter of a future communication.

References

1. N. Distefano, "Nonlinear Processes in Engineering," Academic Press, New York, 1974.
2. N. Distefano and A. Samartin, A dynamic programming approach to the formulation and solution of finite element equations, *Comput. Methods Appl. Mech. Eng.* **5** (1975), pp 37–52.